Derivation of the derivative of the natural logarithm via implicit differentiation

We shall show that $\frac{d}{dx}\left(\ln\left(|x|\right)\right) = \frac{1}{x}$ for all $x \neq 0$ as outlined in the next section.

Approach

- We define $y(x) = \ln\left(|x|\right)$ (1).
- We use the exponential function on (1) and thereby obtain an equation (2).
- We differentiate (2) with respect to x, solve for $\frac{dy}{dx}$ and hence get the desired result.

Derivation

We use the exponential function on both sides of the equation $y(x) = \ln\left(|x|\right) = \begin{cases} \ln\left(x\right) & x > 0 \\ \ln\left(-x\right) & x < 0 \end{cases}$ and get:

$$e^{y(x)} = \begin{cases} e^{\ln(x)} = x & x > 0 \\ e^{\ln(-x)} = -x & x < 0 \end{cases} \qquad (2)$$

We know that $\frac{dx}{dx} = 1$, and also, from one of the definitions of the exponential function, that $\frac{d}{dx}\left(e^{x}\right) = e^{x}$. So, using the chain rule, we can differentiate (2) with respect to x:

$$\frac{d}{dx}\left(e^{y(x)}\right) = \begin{cases} \frac{dx}{dx} & x > 0 \\ -\frac{dx}{dx} & x < 0 \end{cases} \quad \Rightarrow \quad e^{y}\frac{dy}{dx} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$$

We solve for $\frac{dy}{dx}$ and substitute $y(x) = \begin{cases} \ln\left(x\right) & x > 0 \\ \ln\left(-x\right) & x < 0 \end{cases}$:

$$\frac{dy}{dx} = \begin{cases} e^{-y} = e^{-\ln(x)} = \frac{1}{e^{\ln(x)}} = \frac{1}{x} & x > 0 \\ -e^{-y} = -e^{-\ln(-x)} = -\frac{1}{e^{\ln(-x)}} = \frac{1}{x} & x < 0 \end{cases}$$

Thus, $\frac{d}{dx}\left(\ln\left(|x|\right)\right) = \frac{1}{x}$ **for all** $x \neq 0$**.**