Derivation of the derivative of $f(x) = x^s$ via implicit differentiation

For the function $f(x) = x^s$, we shall derive the derivative $f'(x) = \frac{d}{dx}\left(x^s\right) = sx^{s-1}$ for $s \in \mathbb{R}$ as outlined in the next section.

Approach

- We define $y(x) = x^s$ (1).
- We use the absolute value function followed by the natural logarithm on (1) and thereby obtain an equation (2).
- We differentiate (2) with respect to x, solve for $\frac{dy}{dx}$, and thus get the desired result.

Derivation

We use the absolute value function on both sides of (1) and make use of the rule $|x^n| = |x|^n$:

$$|y(x)| = |x^s| = |x|^s$$

Now, we use the natural logarithm on both sides, utilising $\ln\left(x^a\right) = a\ln\left(x\right)$, and we get for all $x \neq 0$:

$$\ln\left(|y|\right) = \ln\left(|x|^s\right) = s\ln\left(|x|\right) \qquad (2)$$

We differentiate (2), using the chain rule and the derivative $\frac{d}{dx}\left(\ln\left(|x|\right)\right) = \frac{1}{x}$, with respect to x:

$$\frac{d}{dx}\left(\ln\left(|y|\right)\right) = s\frac{d}{dx}\left(\ln\left(|x|\right)\right) \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = s\frac{1}{x}$$

We solve for $\frac{dy}{dx}$ and substitute $y = x^s$:

$$\frac{dy}{dx} = s\frac{y}{x} = s\frac{x^s}{x} = \underline{sx^{s-1}}$$

Hence, $\frac{d}{dx}\left(x^s\right) = sx^{s-1}$ except for $x = 0$.

Since, to set up equation (2), we had to insist on $x \neq 0$, we now have to take a closer look at $x = 0$.
A limit definition for the derivative of $f(x)$ at $x = x_0$ is:

$$f'(x_0) = \lim_{x \to x_0}\left(\frac{f(x) - f(x_0)}{x - x_0}\right)$$

So, with $f(x) = x^s$ and $x_0 = 0$, we have:

$$f'(0) = \lim_{x\to 0}\left(\frac{x^s - 0^s}{x-0}\right) = \lim_{x\to 0}\left(\frac{x^s}{x}\right) = \lim_{x\to 0}\left(x^{s-1}\right) = \begin{cases} 0 = \left[sx^{s-1}\right]_{x=0} & s > 1 \\ 1 = \left[sx^{s-1}\right]_{x=0} & s = 1 \\ \text{no solution} & s < 1 \end{cases}$$

Thus, $\frac{d}{dx}\left(x^s\right) = sx^{s-1}$ for $s \in \mathbb{R}$, but there is no solution for the combination $x = 0$ and $s < 1$.