Derivation of Euler's formula via differential equation

We shall derive Euler's formula $e^{ix} = \cos(x) + i\sin(x)$ as described in the next section.

Approach

- We express e^{ix} as a complex number in polar coordinates with radius r and azimuth ϕ (see Figure 1), thus $e^{ix} = r\,(\cos(\phi) + i\sin(\phi))$ (1) .
- We differentiate (1) with respect to x and thereby get an equation (2), wherein we eliminate e^{ix} by using (1).
- We solve (2) for $\frac{dr}{dx}$ and $\frac{d\phi}{dx}$ and therewith find $r(x)$ and $\phi(x)$ by direct integration. We substitute all of these in (2) and obtain an equation (3).
- Using the condition $e^0 = 1$, we find the two constants of integration in (3) and so arrive at the desired result.

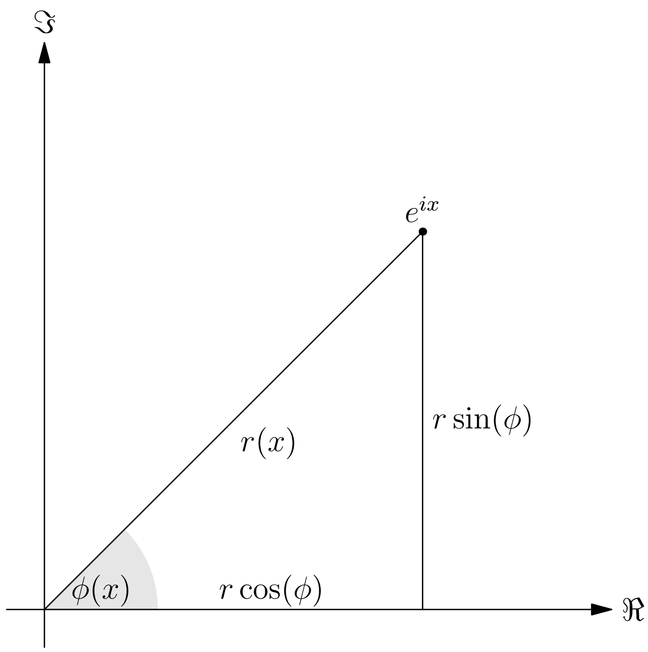

Figure 1: Representation of e^{ix} in polar coordinates.

Derivation

If e^{ix} is a complex number (or a real number, which form a subset of the complex numbers), it must be possible, as with any other complex number, to express it in polar coordinates. This idea leads us directly to equation (1), which we now differentiate with respect to x using the product, chain and constant rules:

$$\frac{d}{dx}\left(e^{ix}\right) = \frac{d}{dx}\left(r(x)\left(\cos(\phi(x)) + i\sin(\phi(x))\right)\right)$$

$$ie^{ix} = \frac{dr}{dx}\left(\cos(\phi) + i\sin(\phi)\right) + r\left(-\sin(\phi)\frac{d\phi}{dx} + i\cos(\phi)\frac{d\phi}{dx}\right) \qquad (2)$$

We substitute e^{ix} on the left-hand side of this equation with the expression in polar coordinates in (1) and expand:

$$ir\left(\cos(\phi)+i\sin(\phi)\right)=\frac{dr}{dx}\left(\cos(\phi)+i\sin(\phi)\right)+r\left(-\sin(\phi)\frac{d\phi}{dx}+i\cos(\phi)\frac{d\phi}{dx}\right)$$

$$ir\cos(\phi)-r\sin(\phi)=\frac{dr}{dx}\cos(\phi)+i\frac{dr}{dx}\sin(\phi)-r\sin(\phi)\frac{d\phi}{dx}+ir\cos(\phi)\frac{d\phi}{dx}$$

Since the real ($\Re$) and imaginary ($\Im$) parts on both sides of this equation must be equal, we obtain the following system of equations:

$$\Re\left(ir\cos(\phi)-r\sin(\phi)\right)=\Re\left(\frac{dr}{dx}\cos(\phi)+i\frac{dr}{dx}\sin(\phi)-r\sin(\phi)\frac{d\phi}{dx}+ir\cos(\phi)\frac{d\phi}{dx}\right)\qquad\text{(I)}$$

$$\Im\left(ir\cos(\phi)-r\sin(\phi)\right)=\Im\left(\frac{dr}{dx}\cos(\phi)+i\frac{dr}{dx}\sin(\phi)-r\sin(\phi)\frac{d\phi}{dx}+ir\cos(\phi)\frac{d\phi}{dx}\right)\qquad\text{(II)}$$

$$-r\sin(\phi)=\frac{dr}{dx}\cos(\phi)-r\sin(\phi)\frac{d\phi}{dx}\qquad\text{(I)}\qquad\qquad r\cos(\phi)=\frac{dr}{dx}\sin(\phi)+r\cos(\phi)\frac{d\phi}{dx}\qquad\text{(II)}$$

We multiply (I) with $\cos(\phi)$ and (II) with $\sin(\phi)$:

$$-r\sin(\phi)\cos(\phi)=\frac{dr}{dx}\cos^2(\phi)-r\sin(\phi)\cos(\phi)\frac{d\phi}{dx}\qquad\text{(Ia)}$$

$$r\sin(\phi)\cos(\phi)=\frac{dr}{dx}\sin^2(\phi)+r\sin(\phi)\cos(\phi)\frac{d\phi}{dx}\qquad\text{(IIa)}$$

We add (Ia) to (IIa) and use the trigonometric identity $\cos^2(\phi)+\sin^2(\phi)=1$:

$$\text{(Ia)+(IIa):}\qquad 0=\frac{dr}{dx}\left(\cos^2(\phi)+\sin^2(\phi)\right)\qquad\Rightarrow\qquad\underline{\frac{dr}{dx}=0}$$

With $\frac{dr}{dx}=0$, (II) becomes:

$$r\cos(\phi)=r\cos(\phi)\frac{d\phi}{dx}\qquad\Rightarrow\qquad\underline{\frac{d\phi}{dx}=1}$$

By direct integration, we find that, with $\frac{dr}{dx}=0$, $\underline{r(x)=c_1}$ and, with $\frac{d\phi}{dx}=1$, $\underline{\phi(x)=x+c_2}$, where c_1 and c_2 are constants of integration. With these, equation (2) becomes:

$$ie^{ix}=c_1\left(-\sin(x+c_2)+i\cos(x+c_2)\right)\qquad\Rightarrow\qquad e^{ix}=c_1\left(\cos(x+c_2)+i\sin(x+c_2)\right)\qquad(3)$$

To find the values of c_1 and c_2, we use the condition $e^0=1$ and hence, with $x=0$, equation (3) is $1=c_1\left(\cos(c_2)+i\sin(c_2)\right)$. Equating real and imaginary parts, as above, gives the following system of equations:

$$\Re\,(1) = \Re\,(c_1 \cos(c_2) + ic_1 \sin(c_2)) \qquad \text{(III)} \qquad\qquad \Im\,(1) = \Im\,(c_1 \cos(c_2) + ic_1 \sin(c_2)) \qquad \text{(IV)}$$

$$1 = c_1 \cos(c_2) \qquad \text{(III)} \qquad\qquad 0 = c_1 \sin(c_2) \qquad \text{(IV)}$$

(IV) is true for $c_1 = 0$ and/or for $\sin(c_2) = 0$. But, c_1 cannot be zero because this would not satisfy (III), thus we need:

$$\sin(c_2) = 0 \qquad \Rightarrow \qquad c_2 = n\pi \text{ for } n \in \mathbb{Z}$$

Since $\cos(c_2) = 1$ for n even, so that $c_2 = 2k\pi$ for $k \in \mathbb{Z}$, and $\cos(c_2) = -1$ for n odd, so that $c_2 = (2k+1)\,\pi$ for $k \in \mathbb{Z}$, we need to analyse both cases, $c_1 = 1$, $c_2 = 2k\pi$ and $c_1 = -1$, $c_2 = (2k+1)\,\pi$ separately.

For $c_1 = 1$, $c_2 = 2k\pi$:

Equation (3) becomes $e^{ix} = \cos(x+2k\pi)+i\sin(x+2k\pi)$. But, since sine and cosine are 2π-periodic, $\cos(x+2k\pi) = \cos(x)$ and $\sin(x+2k\pi) = \sin(x)$ and thus, in this case, we have $e^{ix} = \cos(x) + i\sin(x)$.

For $c_1 = -1$, $c_2 = (2k+1)\,\pi$:

Equation (3) becomes $e^{ix} = -\left(\cos(x + 2k\pi + \pi) + i\sin(x + 2k\pi + \pi)\right)$. But, for the same reason as above, $\cos((x+\pi) + 2k\pi) = \cos(x+\pi)$ and $\sin((x+\pi) + 2k\pi) = \sin(x+\pi)$; and since $\cos(x+\pi) = -\cos(x)$ and $\sin(x+\pi) = -\sin(x)$, we have $\cos(x + (2k+1)\,\pi) = -\cos(x)$ and $\sin(x + (2k+1)\,\pi) = -\sin(x)$. Hence, in this case, we get $e^{ix} = -\left(-\cos(x) - i\sin(x)\right) = \cos(x) + i\sin(x)$ as well.

Both cases lead to the same result, which is $\underline{e^{ix} = \cos(x) + i\sin(x)}$, as expected.