Derivation of the quadratic formula by completing the square

We shall derive the quadratic formula $x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$, which provides the solutions to a general quadratic equation $ax^2 + bx + c = 0$ (1) with $a \neq 0$, as outlined in the next section.

Approach

- We complete the square for (1).
- We solve the equation for the squared binomial.
- We extract the square root and get, by solving the equation thus generated for x, the desired result.

Derivation

We factor out a from (1):

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0$$

After the first two terms in parentheses we insert a constant term $\left(\frac{b}{2a}\right)^2$, which was chosen such that the first three terms in parentheses combine to give a squared binomial. Then, we immediately subtract said term again to not falsify the equation:

$$a\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a}\right) = 0$$

Since $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(x + \frac{b}{2a}\right)^2$, we get:

$$a\left(\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \left(\frac{b}{2a}\right)^2\right) = 0$$

We multiply out the outermost brackets and solve the equation for the squared binomial:

$$a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = 0 \quad \Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}$$

We extract the square root, then solve for x and thereby get the desired result:

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \quad \Rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \underline{\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Thus, the solutions to (1) are given by $x_{1,2} = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$.