Derivation of the Euler-Lagrange equation via Gâteaux differential

We shall derive the Euler-Lagrange equation for the functional $S\left[y(x)\right] = \int_a^b F\left(x, y(x), y'(x)\right)\,dx$ (1) with boundary conditions $y(a) = A$ and $y(b) = B$, namely $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$, $y(a) = A$, $y(b) = B$, as outlined in the next section.

Approach

- If the functional (1) has a stationary path $y(x)$, this path must make the first (generalised) derivative, the Gâteaux differential, of S zero. Thus, we apply the Gâteaux differential to (1), set it to zero and thereby get an equation (2).
- We express $S\left[y + \epsilon h\right]$ in (2) using a Taylor expansion of F about (x, y, y') to $O(\epsilon)$.
- We make use of Leibniz's rule for differentiation under the integral sign to swap the order of operations, and so get an equation (3).
- We eliminate h' in (3) by partial integration, apply the fundamental lemma of the Calculus of Variations to the thus created equation, and, hence, get the desired result.

Derivation

The Gâteaux differential of a functional $S\left[y(x)\right]$ with admissible variation $h(x)$ - so h in $\mathcal{D}_1\left(a, b\right)$ for weak or h in $\mathcal{D}_0\left(a, b\right)$ for strong variations and $h(a) = h(b) = 0$ - and with $F, h \in C^n\left[a, b\right]$ for $n \geq 2$ is:

$$\Delta S\left[y(x), h(x)\right] = \lim_{\epsilon \to 0}\left(\frac{d}{d\epsilon}\left(S\left[y + \epsilon h\right]\right)\right)$$

For a stationary path of the functional S we need:

$$\Delta S = \lim_{\epsilon \to 0}\left(\frac{d}{d\epsilon}\left(S\left[y + \epsilon h\right]\right)\right) = 0 \qquad (2)$$

With an admissible, varied path $y + \epsilon h$, (1) becomes:

$$S\left[y + \epsilon h\right] = \int_a^b F\left(x, y + \epsilon h, y' + \epsilon h'\right)\,dx$$

So, both h and h' must be bounded and, hence, we need h in $\mathcal{D}_1\left(a, b\right)$ to proceed.

A Taylor expansion of F about (x, y, y') to $O(\epsilon)$ gives:

$$F\left(\begin{pmatrix} x \\ y \\ y' \end{pmatrix}^T + \begin{pmatrix} 0 \\ \epsilon h \\ \epsilon h' \end{pmatrix}^T\right) = F\left(x, y, y'\right) + 0 \cdot F_x + \epsilon h F_y + \epsilon h' F_{y'} + O(\epsilon^2)$$

, where $F_x = \frac{\partial F}{\partial x}$, $F_y = \frac{\partial F}{\partial y}$ and $F_{y'} = \frac{\partial F}{\partial y'}$.

Thus, we have:

$$S\left[y+\epsilon h\right]=\int\limits_a^b\left(F\left(x,y,y'\right)+\epsilon hF_y+\epsilon h'F_{y'}+O(\epsilon^2)\right)\,dx$$

And, hence, (2) becomes:

$$\lim_{\epsilon\to 0}\left(\frac{d}{d\epsilon}\left(\int\limits_a^b\left(F\left(x,y,y'\right)+\epsilon hF_y+\epsilon h'F_{y'}+O(\epsilon^2)\right)\,dx\right)\right)=0$$

Since a and b are constants, we can use the special case of Leibniz's rule for differentiation under the integral sign, that is $\frac{d}{dx}\left(\int_a^b f(x,t)\,dt\right)=\int_a^b\frac{\partial}{\partial x}\left(f(x,t)\right)\,dt$:

$$\lim_{\epsilon\to 0}\left(\frac{d}{d\epsilon}\left(\int\limits_a^b\left(F\left(x,y,y'\right)+\epsilon hF_y+\epsilon h'F_{y'}+O(\epsilon^2)\right)\,dx\right)\right)=\lim_{\epsilon\to 0}\left(\int\limits_a^b\frac{\partial}{\partial\epsilon}\left(F\left(x,y,y'\right)+\epsilon hF_y+\epsilon h'F_{y'}+O(\epsilon^2)\right)\,dx\right)=0$$

$$\lim_{\epsilon\to 0}\left(\int\limits_a^b\left(hF_y+h'F_{y'}+O(\epsilon)\right)\,dx\right)=0\quad\Rightarrow\quad\int\limits_a^b\left(hF_y+h'F_{y'}\right)\,dx=0$$

$$\Rightarrow\quad\int\limits_a^b hF_y\,dx+\int\limits_a^b h'F_{y'}\,dx=0\qquad(3)$$

We use integration by parts for the second integral in (3):

Sign	Differentiation	Integration
$+$	$F_{y'}$	h'
-	$\frac{d}{dx}\left(F_{y'}\right)$	h

And, thence, get:

$$\int\limits_a^b h'F_{y'}\,dx=\left[F_{y'}h\right]_a^b-\int\limits_a^b\frac{d}{dx}\left(F_{y'}\right)h\,dx$$

Because (as mentioned above) we need $h(a)=h(b)=0$ for an admissible variation, we have:

$$\left[F_{y'}h\right]_a^b=F_{y'}|_{x=b}h(b)-F_{y'}|_{x=a}h(a)=0$$

Thus, $\int_a^b h'F_{y'}\,dx=-\int_a^b\frac{d}{dx}\left(F_{y'}\right)h\,dx$ and thereby (3) becomes:

$$\int\limits_a^b hF_y\,dx+\int\limits_a^b h'F_{y'}\,dx=\int\limits_a^b hF_y\,dx-\int\limits_a^b\frac{d}{dx}\left(F_{y'}\right)h\,dx=0$$

$$\Rightarrow\quad\int\limits_a^b\left(\frac{d}{dx}\left(F_{y'}\right)-F_y\right)h(x)\,dx=0$$

Now, we apply the fundamental lemma of the Calculus of Variations, whereby $\int_a^b z(x)h(x)\,dx = 0$ for all admissible h only if $z(x) = 0$, and hence:

$$\underline{\frac{d}{dx}\left(F_{y'}\right) - F_y = 0}$$

So, $\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$ with $y(a) = A$ and $y(b) = B$ is necessary for a stationary path $y(x)$ of the functional S. This is the Euler-Lagrange equation for S.