Derivation of the fundamental lemma of the Calculus of Variations

We shall show, as outlined in the next section, that - for $z(x)$ and $h(x)$ continuous in the interval $a \leq x \leq b$ and $h(a) = h(b) = 0$ - if $\int_a^b z(x)h(x)\,dx = 0$ (1) for all admissible $h(x)$, then $z(x) = 0$.

Approach

- We define $h(x) = q(x)z(x)$ and choose $q > 0, \forall x \in (a, b)$ in such a way that h satisfies all the conditions mentioned above.
- We substitute this in (1), and show that, for such a choice of h, (1) can only be satisfied if $z(x) = 0$.

Derivation

Since $h(x)$ in (1) is an arbitrary function (the admissible variation), subject to the conditions $h \in C_0\,(a, b)$ and $h(a) = h(b) = 0$, we can choose $h(x) = q(x)z(x)$ with $q(x)$ continuous in $[a, b]$, $q(a) = q(b) = 0$, and $q > 0, \forall x \in (a, b)$ which (because $z(x)$ is continuous in $[a, b]$ by definition) satisfies these conditions. E.g. $q(x) = -\,(x - a)\,(x - b)$ has roots at $x = a$ and $x = b$, is in $C_0\,(a, b)$, is strictly positive for $x \in (a, b)$ and, hence, is a specific example of such a function.

Substituting such a function in (1) gives:

$$\int\limits_a^b z(x)h(x)\,dx = 0 \quad \Rightarrow \quad \int\limits_a^b q(x)z^2(x)\,dx = 0$$

But, since $q(x) > 0$ and $z(x)^2 \geq 0$ for all $x \in (a, b)$, this equation can only be satisfied if $z = 0$.

Thus, if $\int_a^b z(x)h(x)\,dx = 0$ for all admissible $h(x)$, then we need $z(x) = 0$.